The action of Vlasov waves on the velocity distribution in a plasma

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A one-dimensional model with no magnetic field is considered. It is supposed that the plasma starts in thermal equilibrium and then a current is forced to grow. Instability leads to the growth of waves, which are shown to stir the distribution in phase space, but only over a limited range of velocity. It is concluded that in order to restore stability the energy in the wave must become comparable to the energy of drift.

1. Introduction

Vlasov waves grow when a plasma is electrostatically unstable. This kind of instability may be expected in a great variety of devices designed for confining a plasma and in particular has been suggested as the cause of 'pump out' in Stellarators (Bernstein, Friedman, Kulsrud & Rosenbluth 1960). The existing theory tells us little more than the stability condition in the simplest case, in which the problem is one-dimensional, the plasma is uniform in space and all magnetic effects are ignored. From here it is possible to proceed in various directions by removing different simplifying assumptions, and obviously one should not remove more than one at a time. The aim of this investigation is to remove the assumption of linearization. Linearized theory gives the stability condition, but it is important to know how strong the waves grow and this is an essentially non-linear problem.

The problem is posed as follows. Suppose that a plasma is initially in thermal equilibrium and a gradually increasing current is driven through it by an electric field. If U_D is the difference between the mean velocities of the electrons and ions and v_{te} is the thermal velocity of electrons, Vlasov instability sets in when U_D reaches about $0{\cdot}93v_{te}$. It is evident that the instability takes energy from the electric field, but suppose that the external circuit is such that the current still increases (this is usual in practice). How strong do the Vlasov waves grow and what is their effect on the overall behaviour of the plasma?

If the velocity distribution were always Maxwellian, one could say that the Vlasov waves heat the plasma and the rise in temperature tends to restore stability. The time scale of the Vlasov waves is related to the plasma frequency, which in practice is very large. Typically the plasma period may be 10^{-12} sec, while the time scale T for the growth of the current is seldom less than 10^{-6} sec. Consequently, even if the growth time of the waves were hundreds of plasma periods it would still be short compared with T. This suggests that the waves will adjust their amplitude in the following way. The heating due to the waves should raise the temperature at such a rate that the plasma is always near the critical condition: drift velocity $\approx 0{\cdot}93 \times$ thermal velocity. Then the temperature must rise with a time scale $\sim T$, which suggests that the wave amplitude is not large. If so, the linear theory may be valid, in which case the heating should be obtained from the second-order terms.

In the linear theory the distribution function f, for say the electrons, is written $f^{(0)}+f^{(1)}$, where $f^{(1)}$ is the first-order perturbation. The linearized collision-free Boltzmann equation then gives

$$i(\omega+kv)f^{(1)} = \frac{e}{m}E\,\partial f^{(0)}/\partial v, \tag{1}$$

where ω the angular wave frequency is complex, k the wave number is real and E the electric field is a first-order quantity proportional to $\exp[i(\omega t+kx)]$ (there is no applied field in the linear theory). The heating must depend on the component of the electron current density in phase with the electric field and this is proportional to the imaginary part of

$$\int \frac{v\,\partial f^{(0)}/\partial v\,dv}{\omega+kv} = -\frac{\omega}{k}\int\frac{\partial f^{(0)}/\partial v\,dv}{\omega+kv}.$$

Now it has been suggested that the imaginary part of ω (the growth rate) is small and in this case the imaginary part of the integral comes almost entirely from a small range of velocity near the wave velocity. This is familiar. The growth, or, in the case of Landau damping, decay, is almost entirely due to 'trapped' particles with velocities near the wave velocity. The width of the velocity range involved is of order $k^{-1}\mathscr{I}(\omega)$. Now, because few particles are involved, we can have no confidence in the assumption that the velocity distribution remains Maxwellian, and a deeper investigation is required.

2. The general stability condition

The stability condition for any velocity distribution (uniform in space) has been obtained by several authors and is given in an elegant form by Penrose (1960).

Penrose's analysis also enables us to obtain the velocity and an estimate of the wavelength of the fastest growing wave and his results will now be summarized. They involve the function $F(u)$ defined by

$$F(u) = \sum_j 4\pi e_j^2 f_j(u)/m_j,$$

where $f_j(u)$ is the one-dimensional distribution function for the jth type of particle. The dispersion equation is then

$$k^2 = Z(\omega/k),$$

where

$$Z(\zeta) = \int_{-\infty}^{\infty}(u-\zeta)^{-1}(dF/du)\,du \tag{2}$$

in the upper half-plane and is continued analytically across the real axis.

Penrose represents this by drawing the curve of real ζ on the Z-plane and obtains a necessary and sufficient condition for instability, which is that $F(u)$ should have a minimum at some value ζ of u such that

$$\int_{-\infty}^{\infty} (u-\xi)^{-2}\,[F(u)-F(\xi)]\,du > 0. \tag{3}$$

In the present problem it is easy to see that $F(u)$ already has a minimum when U_D exceeds a small fraction of v_{te} and hence that the onset of instability at $U_D = 0{\cdot}93v_{te}$ is due to the integral in (3) changing from negative to positive.

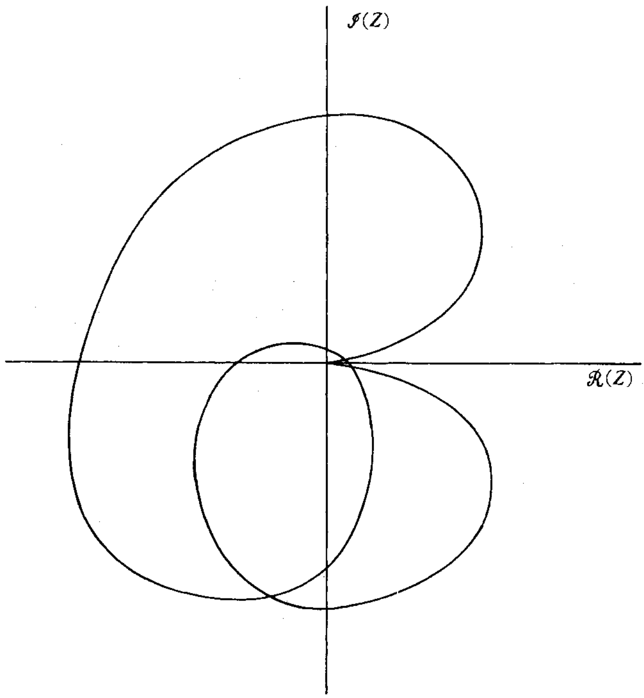

FIGURE 1

When U_D just exceeds the critical value, the curve of real ζ on the Z-plane looks like figure 1, and the minimum $\zeta = \xi$ is the point where the curve intersects the positive real axis. The growing waves are represented by points between here and the origin and, so long as the growth rate is small, satisfy the approximate dispersion relation

$$k^2 - k_0^2 \approx (\zeta-\xi)\,(dZ/d\zeta)_{\zeta=\xi}, \tag{4}$$

where k_0^2 is the value of Z at $\zeta = \xi$. The velocity of the waves is close to ξ and the growth rate is related to the wavelength by

$$\mathscr{I}(\omega) \approx k(k^2-k_0^2)\,\mathscr{I}[(dZ/d\zeta)^{-1}_{\zeta=\xi}].$$

The fastest growing wavelength is given by $k_1 = k_0/\sqrt{3}$, and hence

$$k_1^3 \approx -\tfrac{1}{2}\mathscr{I}(\omega)\{\mathscr{I}[(dZ/d\zeta)^{-1}_{\zeta=\xi}]\}^{-1}. \tag{5}$$

To obtain an order of magnitude for k_1, $\{\mathscr{I}[(dZ/d\zeta)^{-1}_{\zeta=\xi}]\}^{-1}$ may be approximated by $\mathscr{I}[(dZ/d\zeta)_{\zeta=\xi}]$, assuming that $\mathscr{R}(dZ/d\zeta)_{\zeta=\xi}$ is smaller. When ζ is on the upper side of the real axis, we see from (2) that

$$\mathscr{I}(Z) = \pi(dF/du)_{u=\zeta}, \tag{6}$$

and then $$\mathscr{I}(dZ/d\zeta) = \pi(d^2F/du^2)_{u=\zeta}.$$

For a Maxwellian distribution

$$d^2f/dv^2 = \pi^{-\frac{1}{2}}n(m/2kT)^{\frac{3}{2}}(mv^2/kT-1)\exp(-mv^2/2kT).$$

Now at the minimum, mv^2 is the same for protons and electrons and near the critical condition is about $1{\cdot}7kT$. Consequently, the ion contribution dominates in $(d^2F/du^2)_{u=\xi}$ and we find

$$k_1^3 \sim 0{\cdot}3\mathscr{I}(\omega)\,ne^2(\pi^3m_i/2k^3T^3)^{\frac{1}{2}} \sim 5\,.\,10^{-7}\mathscr{I}(\omega)\,nT^{-\frac{3}{2}}\,\text{cm}^{-3}$$

for deuterium.

With $\mathscr{I}(\omega) \sim 10^6\,\text{sec}^{-1}$, $n \sim 10^{16}\,\text{cm}^{-3}$ and $T \sim 10^7\,{}^\circ\text{K}$ this gives a wavelength $2\pi/k_1$ of about 1 mm.

3. Stirring in phase space

We now consider the effect of the wave on the velocity distribution. This may not be very sensitive to the growth rate and, since we are interested in the case where the growth rate is small, it seems sensible to consider the case in which there is no growth or decay, that is the frequency is real. Such waves

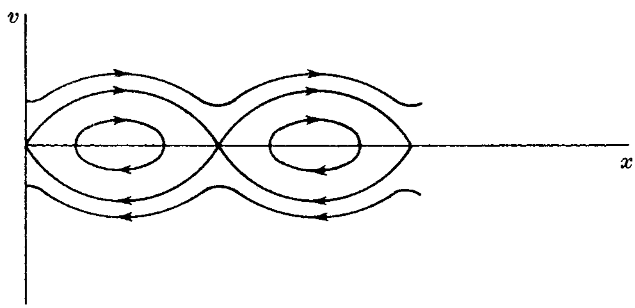

FIGURE 2

have been studied by Bernstein, Greene & Kruskal (1957), who showed that solutions representing waves of finite amplitude and having a considerable degree of arbitrariness exist. Their method is to use a frame moving with the wave and demonstrate the existence of steady solutions. We also use this frame, but study the effect of the electric field belonging to the wave on the distribution function, supposing that the latter is not chosen to give a steady state. In the chosen frame we neglect the variation of the electric field with time and it is then possible to draw the trajectories of particles in phase space. This is done in figure 2 for an electric field $E\cos kx$, which is chosen for simplicity. It is seen that there is a row of 'eddies' with their centres on the zero velocity of this frame. Outside the eddies the trajectories are wavy lines whose amplitude decreases with the mean value of v like v^{-1}, because the frequency seen by a particle with mean velocity v is k/v.

The period of rotation in the centre of an eddy is given by the period of simple harmonic motion in a field $-Ekx$. Hence the angular velocity in the centre of an eddy is

$$S = (eEk/m)^{\frac{1}{2}}. \tag{7}$$

The electrostatic potential is $-k^{-1}E \sin kx$ and the half-width of the eddies in velocity is

$$a = 2(eE/mk)^{\frac{1}{2}} = 2k^{-1}S. \tag{8}$$

This may be compared with the range of velocities of trapped particles found above from the linear theory; when the growth rate is small we may expect a to be the larger.

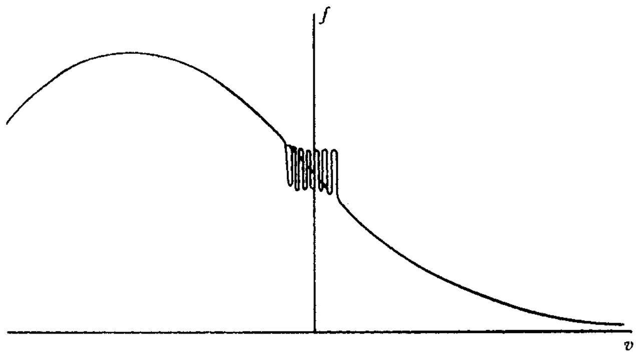

FIGURE 3

Now the collision-free Boltzmann equation is equivalent to Liouville's theorem which states that if one follows the trajectory of a particle in position, velocity and time the distribution function remains constant. Thus in a steady state the contours of constant f must be the trajectories of figure 2. In a non-steady state the motion depicted in figure 2 will 'stir' the distribution and this will be much more effective in the region of the eddies than elsewhere. The effect on the distribution may be seen by supposing that there is a contour of constant f passing through the centres of all the eddies. As the eddies rotate this will be wound into spirals, one in each eddy. Each rotation of an eddy adds a turn to the spiral and hence the spiral becomes tighter. The effect on the distribution function at a position corresponding to the centre of an eddy (that is a potential trough) is shown in figure 3. The dotted line represents the unperturbed distribution function, and the full line shows the rapid variations caused by the stirring; each turn of the eddy increases the numbers of maxima and minima in this curve by two. It then follows that the scale of variation of f with velocity in the eddies becomes smaller and hence that $|\partial^2 f/\partial v^2|$ becomes larger. This shows that collisions must be taken into account and the Fokker–Planck equation is appropriate. The full Fokker–Planck expression (Rosenbluth, MacDonald & Judd 1957) is complicated, but for a preliminary study we need only note that it has the form

$$\frac{\partial}{\partial v}\left[G(v)\frac{\partial f}{\partial v} + H(v) f\right],$$

where G and H depend on integrals over v involving f, but these are unlikely to be greatly altered by the sort of disturbance shown in figure 3. Thus the Fokker–

Planck term may be described as representing diffusion of particles in velocity, the stable distribution function towards which they tend to diffuse being the Maxwellian distribution determined by the number of particles and their total momentum and energy. Like a diffusion term it removes small-scale variations more quickly than large-scale variations. As $|\partial^2 f/\partial v^2|$ is increased by the stirring the Fokker–Planck term must eventually become important. When it smooths out the oscillations in figure 3, the full curve of figure 4 results. This then gives a qualitative idea of the effect of the wave on the distribution function: the distribu-

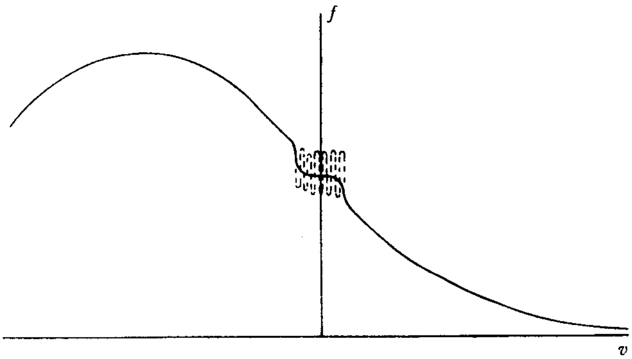

FIGURE 4

tion function develops a ledge centred on the wave velocity. Clearly particles have been transported across the wave velocity from the side of high f to that of low f. When the wave velocity is at the minimum of F, this reduces the current density as expected. The stirring mechanism is similar to a Fermi mechanism in that it tends to produce a flat distribution in the velocity range in which it operates. The effect on the velocity distribution tends to make it more stable, but it has been assumed that the current is forced to increase by the external circuit.

4. The mechanism of stabilization

The Penrose criterion is strictly applicable only when the distribution functions are uniform in space. It may be used to discuss the development of the waves, however, by formulating the problem as 'if the waves stabilize themselves and die away and the spatial variation of the distribution is smeared out, does a new wave grow?' On this basis the Penrose criterion will be applied to distributions in which ledges have developed as a result of the waves' action.

The difference between the effect of the waves on the electron and proton distributions can be seen by noting that S and a are both proportional to $m^{-\frac{1}{2}}$. For a wave at the velocity of the original minimum the two distributions can be scaled, all velocities being smaller for the ions in the ratio $(m_e/m_i)^{\frac{1}{2}}$ and times being longer for the ions in the ratio $(m_i/m_e)^{\frac{1}{2}}$. The collision time also behaves in this way and hence the Fokker–Planck term also scales. Because the ledge developed in the electron distribution is wider than that of the ions, the minimum in the new distribution is at the inner edge of the electron ledge, that is at a velocity nearer the mean electron velocity than the original minimum. The electric field driving the current also moves the electron distribution in this direction, and only moves the ion distribution by a fraction m_e/m_i as much. Consequently, one expects the minimum to move away from the mean ion velocity, and one may expect from Penrose's criterion that the velocity of the wave U_w will follow that of the minimum. As U_w changes, the eddies in figure 2 sweep across the distribution and churn particles down the slope. If U_w could sweep far enough across the electron distribution, it would be possible by means of a succession of weak waves to stabilize the plasma for a given current density; this has been verified by computations. It is unreasonable that the minimum can move very far from the mean ion velocity, however. On the ledge formed by the action of the wave $\partial f_e/\partial v$ is small, but not very small; even though f_e is very nearly constant in the centre of an eddy, there must be an appreciable variation between the eddies which contributes to the space average. In the tail of the Maxwell distribution for ions, however, $\partial f_i/\partial v$ decreases very rapidly and so the minimum cannot be far out in their tail; more precisely the difference between the velocity at the minimum and the mean velocity of the ions cannot be many times v_{ti} unless the ion tail itself were spread by the wave sufficiently, but it has already been seen that the effect of the wave on the ions is much less than its effect on the electrons.

The argument based on figure 1 that U_w must be near the velocity of minimum F breaks down when F is nearly constant over a range of velocities including the minimum, which is the case after the formation of a ledge. Then from (6), $\mathscr{I}(Z)$ for real ζ is small over the whole range of velocity in which F is nearly constant. It is then possible for the fastest growing wave to have its velocity anywhere in this range and, if U_w were near the end of the range nearest the electrons, stabilization by weak waves would be possible. In order for this to happen $\mathscr{R}(Z)$ for U_w must be less than $\mathscr{R}(Z)$ for ξ but still positive. With a Maxwellian distribution distorted by a small ledge, however, one finds that $\mathscr{R}(Z)$ increases as ξ moves towards the electrons, and consequently U_w can still not be far from the ion velocity. When the current density grows to several times its original critical value, it is necessary to redistribute the velocities of a large proportion of the electrons in order to restore stability. This requires the trapping of a large fraction of the electrons and hence the potential difference in the wave must be comparable to the energy of relative motion $\sim \frac{1}{2}m_e U_D^2$; the energy of the wave is also of this order. With the numerical values given previously the rate of stirring S is then $\sim 5.10^{10}\,\text{sec}^{-1}$ for electrons and $\sim 10^9\,\text{sec}^{-1}$ for deuterons. Thus the stirring process, which has been assumed to form ledges in the distributions, has plenty of time to occur.

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